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# Existences solution of Nonlinear Functional Integral Equation of Fractional order

**P.M.More[1] Dr. B.D. Karande[2]**
**Department of Mathematics**
*B.S.S. ASC,College Makni.Maharashtra*
*Maharashtra Udayagiri Mahavidyalaya Udgir Maharashtra*

**Abstract**

In this we study Existences solution of Nonlinear Functional integral equation of fractional order Solution in Banach space of real function defined continuous bounded on an unbounded intervals by using Schunder Fixed point theorem.

**Keyword**

Fractional order**,** nonlinear functional integral equation, Fixed point theory, Locally Attracativity, Banach Space.

## Introduction



Nonlinear quadratic integral equations appear very often, in many applications of real world problem. For examples, quadratic integral equations are often applicable in the theory of radioactive transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The theory of integral equations of fractional order has recently received a lot of attention and constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs devoted to differential and integral equations of fractional order have appeared (see [1-3, 5-7, and 9-17].In recent years, different and integral equation of fractional order have found wide applications in physics mechanics, engineering, electro chemistry, economics and other fields [15,16,19,23,25].A lot of papers have been devoted to the problem of existence of solutions of nonlinear differential and integral equations of fractional order [1,10-12,14,18,24] However, only a few papers appeared on the existence and properties of solutions of functional integral or differential equations of fractional order on an unbounded interval[6-8,22].
Consider the following functional integral equation order with deviating arguments:

$$x(t) = g\big(t, x(\eta(t))\big) + \frac{f(t,x(\beta(t)))}{\Gamma(\alpha)} \int_0^t \frac{h(t,s)u(s,x(\gamma(s)))}{(t-s)^{1-x}}\, ds, \qquad (1.1)$$

Where $t \epsilon R_+ = [0,\infty], \alpha \in (0,1)$ is a fixed number and $\Gamma\,(\alpha)$ denotes the gamma function.
The aim of this paper is to study the existence of solutions of a nonlinear functional integral equation of fractional order with deviating arguments in the space of real functions, continuous and bounded on an unbounded interval. The technique used here is the measure of non compactness associated with the Scheduler fixed point theorem. Moreover, we will investigate an important property of the solutions which is called the local attractivity of solutions. This property is a generalization of the global attractively of solutions introduced in [17] and is also a variant of the property of asymptotic stability of solutions considered in [2-5,13, 20,21]. Eq. (1.1) studied in the paper is a generalization of Chandrasekher type equations [9].obtained in this paper generalize several ones obtained earlier by many authors.

## 2. Preliminaries

First we recall a few facts concerning fractional calculus[23]. Denote by L' (a, b) the space of real functions defined and Labesgue integrable on the interval(a, b),which is equipped with the standard norm. Let$x \in L^{'}(a,b)$ and let $\propto > 0$ be fixed number[23-31].The Riemann-Liouville fractional integral of order $\propto$Of the function $x(t)$ is defined by the formula

$$I^{\propto}x(t) = 1/\Gamma(\propto) \int_a^t \frac{x(s)}{(t-s)^{1-\propto}} ds, t \in (a,b),$$

Where $r(\propto)$denotes the gamma function.it may be shown that the fractional integral operator$I^{x}$ transfroms the space $L^1(a,b)$ into itself and has some other properties [7, 20, 23,25].
Next we present some facts concerning the measures of non compactness [5].
Suppose E is a real Banach space with the norm $\|.\|$ and the zero element $\theta$.Denote by $B(x,r)$ the closed ball centered at $x$ and with radius r. We write $B_r$ for the ball $B(\theta,r)$. If X is a subset of E then the symbole $\bar{X}$ and Conv. X stand for the closure and convex closure of X, respectively[9-23]. Further, let denote the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.
We define the following notion of measure of non compactness.

## **Definition** 2.1

A mapping $\mu: \mathcal{N}_E \to R_+$ is said to be a measure of non compactness in the space E if it satisfies the following conditios:
(i)The family ker $\mu = \{X\epsilon\mathcal{N}_E: \mu(X) = 0\}$ is nonempty and ker$\mu = \{X\epsilon\mathcal{N}_E: \mu(X) = 0\}$is nonempty and ker$\mu\epsilon\mathcal{N}_E$;
(ii)$X \in Y \Longrightarrow \mu(X) \le \mu(Y)$;
(iii)$\mu(\overline{X}) = \mu(ConvX) = \mu(X)$;
(iv) $\mu(\lambda X + (1-\lambda)Y \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda\epsilon[0,1]$;
(v)If ($X_n$)is a sequence of closed sets from $\mu_E$ such that $X_{n+1} \in X_n$ for n=1,2,3….. and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = n_{n-1}{}^{\infty}X_n$ nonempty.
The family ker $\mu$ defined in axiom (i)is called the kernel of the measure of non compactness $\mu$

**Remarks2.1.**

Let us mention that the intersection set $X_\infty$from (v)is a member of the kernel of the measure of noncompactness $u$.indeed,from the inequality$u(X_\infty) \le u(X_n)$for n=1,2……..we infer that $u(X_\infty = 0$ so $X_\infty\epsilon$ ker $u$.This property of of the intersection set $X_\infty$ will be crucial in our study.Further facts concerning measures of noncompactness and their properties may be found in [5,7].
Now we will work in the Banach space BC($R_+$) consisting of all real functions defined, continuous and bounded on $R_+$with the norm $\| x \| = sup\{|x(t)|: t \ge 0\}$
Will use a measure of compactness in the space BC($R_+$)which was introduced in [5]. In order to define this measure let us fix a nonempty bounded subset X of the space nd a positive number BC ($R_+$) and a positive number T. For $x\epsilon X$ and $\in \ge 0$ denote by $\omega^T(x,\epsilon)$the modules of continuity of the function x on the interval[0,T]i.e.,

$$\omega^T(x,\epsilon) = \sup\{|x(t) - x(s)|: t, s\epsilon[0,T], |t-s| \le \in\}$$

Further, let us put

$$\omega^T(x,\epsilon) = \sup\{\, \omega^T(x,\epsilon) : x\, \epsilon X\}$$
$$\omega_0{}^T(X) = \lim_{\in\to 0} \omega^T(X,\epsilon),$$
$$\omega_0(X) = \lim_{T\to\infty} \omega_0{}^T(X).$$

If t is fixed number from $R_+$ let us denote $X(t) = \{x(t): x \in X\}$ and
$diamX(t) = \sup\{|x(t) - x(t): x, y \in X|\},$

Finally consider the function $u$ defined on the family$\mathcal{N}$BC(R+) by the formula

$$\mu(X) = \omega_0(X) + \lim_{t\to\infty} \sup diamX(t). \qquad (2.1)$$

It may be shown that the function $\mu$is the measure of non compactness in the space BC(R+).The kernel ker $\mu$is the family all nonempty and bounded sets X such that functions belonging to X are locally equicontinuous on R+ and the thickness of the bundle formed by functions from X tends to zero at infinity. This property will permit us to characterize solutions of integral equation.

Now let us assume that is non empty subset of the space BC(R+) and Q is open at Defined on Ψ with values in BC(R+)[6-8].

Consider the following operator equation:

$$x(t) = (Qx)(t), t \geq 0 \qquad (2.2)$$

## Definition 2.2

We say that the solutions of eq.(2.2)are locally attractive if there exists a closed ball B(X$_0$,r)in the space BC(R+)such that for arbitrary solutions $x = x(t)$and $y = y(t)$of eq (2.2) belonging to $B(x_0, r) \cap \Psi$we have that

$$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0 \qquad (2.3)$$

In the case when the limit (2.3) is uniform with respect to the set $B(x_0, r) \cap \Psi$ i.e.when for each $\in > 0$ there exists T>0 such that

$$|x(t) - y(t)| \leq \in \qquad (2.4)$$

For all $x.y \in B(x_0, r) \cap \Psi$ beining solutions of eq.(2.2) and for $t \geq T$,we will say that solutions of eq.(2.2)are uniformly locally attractive(or equivalently ,that solutions of eq.(2.2)are asymptotically stable

## 3. Existence Results

In this section we prove the main result of the paper ,for that we assume the following assumptions.

(A$_1$). The function $g: R_+ * R \to R$ is continuous and there exists a constant $p \geq 0$such that

$$|g(t,x) - g(t,y)| \leq p|x - y|$$

For any $t\epsilon R_+$ and for all $x, y \in R$.

(A$_2$). The function $f: R_+ * R \to R$ is continuous and there exists a constant $q \geq 0$such that

$$|f(t,x) - f(t,y)| \leq q|x - y|$$

For any $t\epsilon R_+$ and for all $x, y \in R$.

(A$_3$). The function$\eta, \beta, \gamma: R_+ \to R_+$ are continuous, and $\eta(t) \to \infty, \beta(t) \to \infty as\ t \to \infty$.

(A$_4$). The function $h: R_+ * R \to R$ is continuous and there exists function $a, b: R_+ \to R_+ b$eing continuous $R_+$such that $|h(t,s)| \leq a(t)b(s)$ For any $, s\ \epsilon R_+$ .

(A$_5$) The function $u: R_+ * R \to R$ is continuous moreover,there exists a function $\emptyset: R_+ \to R_+$ being continuous and nondecreasing on $R_+$ a and a constant k$\geq$ 0such that $|u(t,x)| \leq k\emptyset(|x|)$

For any $t\epsilon R_+$ and for all $x \in R$.

Now denote by $\bar{G}$ and $\bar{F}$ the following constants: $\overline{G} = \sup\{|g(t,0)|: t\ \epsilon R_+\}$, and

$\bar{F} = \sup\{|f(t,0)|: t\ \epsilon R_+\}$

Obviously,$\bar{G}$,, $\bar{F} < \infty$ in view of assumptions (H1)and (H2).Further,let us denote by $\bar{b}$(t)the function defined on $R_+$in the following way: $\bar{b}$ (t)=$\int_0^t \frac{b(s)}{(t-s)^{1-\alpha}} ds$

It is easily seen that$\bar{b}$ (t) continuous on$R_+$.

(A$_6$). The function$\bar{n}$:$R_+ \to R_+$ defined by the formula , $\bar{n}\,(t) =$a (t)$\overline{b}$ (t),its bounded on $R_+$and $\lim_{t\to\infty} \bar{n}\,(t) = 0$

Keeping in mind, the above assumption we define the following constant:$\overline{N} = \sup\{\bar{n}\,(t): t\ \epsilon R_+\}$

$N = \overline{N}$ N and M=k $\bar{F}\,\overline{N}$.

Now, we formulate our last assumption:

($A_7$) there exists a positive solution $r_0$ of the inequality

$$pr + \bar{G} + \frac{1}{\Gamma(\alpha)}[\mathrm{Nr}]\emptyset(r) + M\emptyset(r)] \leq r, \qquad \text{Such that} \qquad (p + \frac{N\emptyset(r^0)}{\Gamma(\propto)}) < 1.$$

**Theorem3.1.** Under assumption ($A_1$)-($A_7$),Eq.(1.1)has at least one solution $x = x(t)$ which belongs to the space BC($R_+$).Moreover, the solution of eq.(1.1)are uniformly locally attractive on $R_+$.
**Proof.** Consider the operator V defined on the space BC($R_+$).by the formula

$$(V_x)(t) = g\left(t.x(\eta(t))\right) + \frac{f(t,x(\beta(t)))}{r(\alpha)}\int_0^t \frac{h(t,s)u(s,x(\Gamma(s)))}{(t-s)^{1-x}}ds. \tag{3.1}$$

in order to simplify our considerations we represent the operator V in the form

$$(V_x)(t) = (G_x)(t) + (F_x)(t) \tag{3.2}$$

Where

$$(G_x)(t) = g\left(t.x(\eta(t))\right),$$
$$(F_x)(t) = f\left(t.x(\beta(t))\right),$$
$$(U_x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{h(t,s)u(s,x(\Gamma(s)))}{(t-s)^{1-\propto}}ds$$

Observe that in view of our assumptions, for any function $x\epsilon BC(R_+)$ the function $G_x$ and $F_x$ are continuous on $R_+$. We show that the same holds also for the function $U_x$. To do this fix T > 0. ∈>0.
Next assume that $t_1.t_2 \in [0,T]$ are such that $|t_2 - t_1| \leq \in$.Without loss of generality we can assume that $t_1<t_2$. Then, in view of imposed assumptions, we have

$$|(U_x)(t_2) - (U_x)(t_1)| \leq \frac{1}{r(\alpha)}\int_0^{t_2}\frac{h(t_2.s)u\left(s,x(\Gamma(s))\right)}{(t_2-s)^{1-\propto}}ds - \int_0^{t_1}\frac{h(t_2,s)u\left(s,x(\Gamma(s))\right)}{(t_2-s)^{1-\propto}}ds\int_0^{t_1}\frac{h(t_2,s)u\left(s,x(\Gamma(s))\right)}{(t_2-s)^{1-\propto}}ds$$
$$-- \int_0^{t_1}\frac{h(t_1,s)u\left(s,x(\Gamma(s))\right)}{(t_1-s)^{1-\propto}}ds \leq \left|\frac{1}{r(\alpha)}\right|\int_0^{t_2}\frac{h(t_2,s)u\left(s,x(\Gamma(s))\right)}{(t_2-s)^{1-\propto}}ds + \int_0^{t_1}\frac{h(t_2,s)u\left(s,x(\Gamma(s))\right)}{(t_2-s)^{1-\propto}}ds$$
$$-\int_0^{t_1}\frac{h(t_1.s)u\left(s.x(\Gamma(s))\right)}{(t_2-s)^{1-\propto}}ds + \int_0^{t_1}\frac{h(t_1s)u\left(s.x(\Gamma(s))\right)}{(t_2-s)^{1-\propto}}ds + \int_0^{t_1}\frac{h(t_1.s)u\left(s.x(\Gamma(s))\right)}{(t_1-s)^{1-\propto}}ds$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\left|h(t_2.s)u\left(s,x(\Gamma(s))\right)\right|}{(t_2-s)^{1-\propto}}ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{h(t_2.s)u\left(s,x(\Gamma(s))\right)}{(t_2-s)^{1-\propto}}ds$$
$$+\frac{1}{r(\alpha)}\int_0^{t_1}|h(t_1.s)|\left|u\left(s,x(\Gamma(s))\right)\right|\left|\frac{1}{(t_2-s)^{1-\propto}} - \frac{1}{(t_1-s)^{1-\propto}}\right|ds \leq \frac{ka(t_2)}{\Gamma(\alpha)}\int_{t_2}^{t_1}\frac{b(s)\Phi(|x(\Upsilon(s))|)}{(t_2-s)^{1-\propto}}$$
$$+\frac{k\omega^T(h,\epsilon)}{\Gamma(\alpha}\int_0^{t_1}\frac{\phi(|x(\Upsilon(s))|}{(t_2-s)^{1-\propto}}ds + \frac{ka(t_1)}{\Gamma(\alpha)}\int_0^{t_1}b(s)\phi(|x(\Upsilon(s))|\left[\frac{1}{(t_1-s)^{1-\propto}} - \frac{1}{(t_2-s)^{1-\propto}}\right]ds$$
$$\leq \frac{ka_Tb_T\Phi(\|x\|)}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{1}{(t_2-s)^{1-\propto}}ds + \frac{ka(t_1)}{\Gamma(\alpha)}\int_0^{t_1}\frac{1}{(t_2-s)^{1-\propto}} + \frac{ka_Tb_T\Phi(\|x\|)}{\Gamma(\alpha)}$$
$$\times\int_0^{t_1}\left[\frac{1}{(t_1-s)^{1-\propto}} - \frac{1}{(t_2-s)^{1-\propto}}\right]ds \leq \frac{ka_Tb_T\Phi(\|x\|)}{\Gamma(\alpha+1)}(t_2-t_1)^{\propto} + \frac{ka_Tb_T\Phi(\|x\|)}{\Gamma(\alpha+1)}[t_2^{\propto} - (t_2-t_1)^{\propto}]$$
$$+\frac{ka_Tb_T\Phi(\|x\|)}{\Gamma(\alpha+1)}[t_1^{\propto} - t_2^{\propto} + (t_2-t_1)^{\propto}] \leq \frac{2ka_Tb_T\Phi(\|x\|)}{\Gamma(\alpha+1)}(t_2-t_1)^{\propto} + \frac{k\omega^T(h,\epsilon)\Phi(\|x\|)}{\Gamma(\alpha+1)}t_2^{\propto} \tag{3.3}$$

Where we denote

$$a_T = max\{a(t): t \in [0,T]\}, b_T = max\{b(t): t \in [0,T]\},$$
$$\omega^T(h,\epsilon) = \sup\{|h(t_2,s) - h(t_1,s)|: s,t_1,t_2 \in [0,T], |t_2 - t_1| \leq \in\}$$

Observe that invoking the uniform continuity of the function h (t,s)on the set[0,T]*[0,T]we deduce that $\omega^T(h,\epsilon) \to 0$ as $\in \to 0$ Further, Keeping in mind the estimate(3.3) we obtain

$$\omega^T(U_x \in) \leq \frac{1}{\Gamma(\alpha+1)}[2ka_Tb_T\phi(||x||)]T^{\alpha} \tag{3.4}$$

Linking inequality (3.4) with the established facts we conclude that the function $U_x$ is continuous on the interval [0,T] for any T > 0.This yields the continuity of the function $U_X$ is continuous on the interval [0,T]for any T > 0. This yields the continuity of $U_x$ on $R_+$.
Finally, combining the continuity of the functions $A_x$, $F_x$ and $U_x$ we deduce that the function $V_x$ is continuous on $R_+$.
Now, taking a function $x \in BC(R_+)$.for an arbitrarily $t \in R_+$.we get

$$|(V_x)(t)| \leq \left|(G_x)(t) + |(F_x)(t)|\right|\left((U_x)(t)\right|$$

$$\leq |g(t.x(\eta(t)) - g(t,0)| + |g(t,0)| + \frac{1}{\Gamma(\alpha)}[|f\left(t, x(\beta(t))\right) - f(t,0)|]\int_0^t \frac{|h(t,s)||\, u(s,x(\gamma(s)))|}{(t-s)^{1-\alpha}}ds$$

$$\leq p|x(x(\eta(t)) + \bar{G} + \frac{ka(t)[q|x(\beta(t)| + \bar{F}]}{\Gamma(\alpha)}\int_0^t \frac{b(s)\phi(|x(\gamma(s)|)}{(t-s)^{1-\alpha}}ds$$

$$\leq p\|x\| + \bar{G} + \frac{ka(t)\|x\|\Phi(\|x\|)}{\Gamma(\alpha)}\int_0^t \frac{b(s)}{(t-s)^{1-\alpha}}ds + \frac{k\bar{F}a(t)\phi(\|x\|)}{\Gamma(\alpha)}\int_0^t \frac{b(s)}{(t-s)^{1-x}}ds$$

$$\leq p\|x\| + \bar{G} + \frac{ka(t)\bar{b}(t)}{\Gamma(\alpha)}\|x\|\Phi(\|x\|) + \frac{kfa(t)\bar{b}(t)}{\Gamma(\alpha)}\Phi(\|x\|) \leq p\|x\| + \bar{G} + \frac{kq\bar{n}(t)}{\Gamma(\alpha)}\Phi(\|x\|) \qquad (3.5)$$

Now, keeping the assumptions, estimate (3.5) yields

$$\|V_x\| \leq p\|x\| + \bar{G} + \frac{ka\bar{N}}{\Gamma(\alpha)}\|x\|\Phi(\|x\|) + \frac{k\bar{F}\bar{N}}{\Gamma(\alpha)}\Phi\left(\|x\| \leq p\|x\| + \bar{G} + \frac{1}{\Gamma(\alpha)}[N\|x\|\phi\|x\|) + M\phi(\|x\|)\right) \qquad (3.6)$$

Combining this estimate with our assumptions we deduce that there exists a number $r_0>0$ such that the operator V transforms the ball$B_{r0}$into itself.
Now let us take a nonempty subset $X \in B_{r0}$then for x, y ∈ X and for an arbitrarily fixed $t \in r_+$we have

$$|(Vx)(t) - (Vy)(t)|$$
$$\leq \left|g\left(t.x(\eta(t))\right) - g\left(t.y(\eta(t))\right)\right|$$
$$+ \left|\frac{f\left(t.x(\beta(t))\right)}{\Gamma(\alpha)}\int_0^t \frac{h(t,s)u\left(s.x(\gamma(s))\right)}{(t-s)^{1-\alpha}}ds - \frac{f\left(t.y(\beta(t))\right)}{\Gamma(\alpha)}\right|$$
$$\times \int_0^t \frac{h(t,s)u\left(s,y(\gamma(s))\right)}{(t-s)^{1-\alpha}}ds| \leq p|x(\eta(t) - y(\eta(t)| + \frac{1}{\Gamma(\alpha)}|f\left(t,x(\beta(t))\right)$$
$$- f(t,y(\beta(t)))$$
$$\times \int_0^t \frac{h(t,s)u\left(s,x(\gamma(s))\right)}{(t-s)^{1-\alpha}}ds + \frac{f\left(t.y(\beta(t))\right)}{\Gamma(\alpha)}\int_0^t \frac{h(t,s)u\left(s.x(\gamma(s))\right) - u\left(s,y(\gamma(s))\right)}{(t-s)^{1-\alpha}}ds$$

$$\leq p\left|x(\eta(t)) - y(\eta(t))\right| + \frac{ka(t)|x(\beta(t))|}{\Gamma(\alpha)}\int_0^t \frac{b(s)\left|u\left(s.x(\gamma(s))\right)\right|}{(t-s)^{1-\alpha}}ds$$
$$\frac{+a(t)[\left|f\left(t,y(\beta(t))\right) - f(t,0)\right| + |f(t,0)|]}{\Gamma(\alpha)}\int_0^t \frac{b(s)\left|u\left(s.x(\gamma(s))\right)\right| + u\left(s.y(\gamma(s))\right)}{(t-s)^{1-\alpha}}$$
$$\leq p\left|x(\eta(t)) - y(\eta(t))\right| + \frac{ka(t)\left|x(\beta(t))\right| - y(\beta(t))|}{\Gamma(\alpha)}\int_0^t \frac{b(s)\Phi(|x(\gamma(s))|)}{(t-s)^{1-\alpha}}ds$$
$$+ \frac{ka(t)[q|y(\beta(t)) + |f(t,0)|]}{\Gamma(\alpha)}$$

$$\times\int_0^t \frac{b(s)\Phi(|x(\gamma(s))|)+\Phi(|y(y(s))|]}{(t-s)^{1-\alpha}}ds \le p|x(\eta(t)-y(\eta(t))| + \frac{kqa(t)\bar{b}\,(t)\Phi(\mathrm{r}_0)}{\Gamma(\alpha)}|x(\beta(t)) - y(\beta(t)) + \frac{2qa(t)\bar{b}\,(t)}{\Gamma(\alpha)}\mathrm{r}_0\Phi\mathrm{r}_0 + \frac{2kfa(t)\bar{b}(t)}{\Gamma(\alpha)}\Phi\mathrm{r}_0 \le \mathrm{pdiamX}(\eta(t)) + \frac{kq\overline{N}\,\Phi(\mathrm{r}_0)}{\Gamma(\alpha)}|diamX(\beta(t)) + \frac{2k\bar{n}(t)}{\Gamma(\alpha)}\mathrm{r}_0\Phi\mathrm{r}_0 + \frac{2\mathrm{k}\bar{\mathrm{n}}(\mathrm{t})}{\Gamma(\alpha)}\Phi\mathrm{r}_0,$$

From the above estimate, we derivative the following inequality:

$$\mathrm{Diam(VX)(t)} \le \mathrm{pdiamX}(\eta(\mathrm{t}) + \frac{N\Phi(\mathrm{r}_0)}{\Gamma(\alpha)}diamX(\beta(t)) + \frac{2k\bar{n}t}{\Gamma(\alpha)}\mathrm{r}_0\Phi\mathrm{r}_0 + \frac{2k\bar{n}t}{\Gamma(\alpha)})\Phi\mathrm{r}_0$$

Hence, from assumption (H6) we get

$$\lim_{t-\infty}\sup diam(VX)\,(t) \le k\lim_{t-\infty}\sup diam(X)\,(t), \quad (3.7)$$

Where$k = p + \frac{N\Phi\mathrm{r}_0}{\Gamma(\alpha)}$

Obviously in view of assumption ($A_7$) we have that k<1.

Further, let us take arbitrary numbers T>0and $\epsilon > 0$. Next, fix arbitrarily a function $x\epsilon X$and $t_1 t_2\epsilon[0,T]$such that $|t_2 - t_1| \le \epsilon$with out loss of generally we may assume that $t_1 < t_2$ then, taking into account our assumptions and using the previously obtained estimate (3.3) we get

$$|(Vx)(t_2) - (Vx)(t_1) \le |(Gx)(t_2) - (Gx)(t_1)| + |(F_X)(t_2)(U_X)(t_2) - (F_X)(t_1)(U_X)(t_2) - (F_X)(t_1)(U_X)(t_1)$$

$$\le \left|g\left(t_2, x(\eta(t_2))\right) - g\left(t_1, x(\eta(t_1))\right)\right| + \frac{\left|f\left(t_{2,}x(\beta(t_2))\right) - f\left(t_{1,}x(\beta(t_1))\right)\right|}{\Gamma(\alpha)}\int_0^{t_2}\frac{|h(t_2,s)|\,|u(\gamma(s)))|}{(t_2-s)^{1-\alpha}}ds + \frac{|f(t_1.x(\beta(t_1)))}{\Gamma(\alpha+1)}\left[2ka_T b_T\phi(||x||)\epsilon^x + k\omega^T(h,\epsilon)\phi(||x||)T^\alpha\right]$$

$$\le |g(t_2,x(\eta(t_2))) - g(t_2,x(\eta(t_1)))| + |g(t_2,x(\eta(t_1))) - g(t_{1,}x(\eta(t_1)) + \frac{|f(t_2,x(\beta(t_2))) - f(t_2,x(\beta(t_1)))| + |f(t_2,x(\beta(t_1))) - f(t_1,x(\beta(t_1)))}{\Gamma(\alpha)}$$

$$\times\int_0^{\mathrm{t}_2}\frac{\mathrm{ka(t_2)b(s)\phi(|x(\gamma(s)))|}}{(\mathrm{t}_2-\mathrm{s})^{1-\alpha}}\mathrm{ds} + \frac{|\mathrm{f(t_{1,}x(\beta(t_1))) - f(t,0)| + |f(t,0)|}}{\Gamma(\alpha+1)}\left[2ka_t b_t\phi(||x||)\epsilon^x + k\omega^T(h,\epsilon)\phi(||x||)T^\alpha\right])$$

$$\le \mathrm{p|x(\eta(t_2)x(\eta(t_1))|}\Psi^{\mathrm{T}}(\mathrm{g}.\epsilon)\frac{\mathrm{ka(t_2)[q|x(\beta(t_2) - x(\beta(t_1))| + \Psi^T(f,\epsilon)}}{\Gamma(\alpha)}\int_0^{\mathrm{t}_2}\frac{\mathrm{b(s)\phi(|x(\gamma(s))|)}}{(\mathrm{t}_2-\mathrm{s})^{1-\alpha}}\mathrm{ds} + \frac{\mathrm{q|x(\beta(t_1))| + |f(t.0)|}}{\Gamma(\alpha+1)}[[2ka_t b_t\phi(||x||)\epsilon^x + k\omega^T(h,\epsilon)\phi(||x||)T^\alpha]\mathrm{p}\Psi^{\mathrm{T}}(\mathrm{x},\mathrm{v}^{\mathrm{T}}(\eta,\epsilon) + \omega^T(g,\epsilon)$$

Further let us consider the sequence $({B^n}_{r0})$, where${B^1}_{r0} = ConvV(B^1_{r0}), {B^1}_{r0} = ConvV(B^1_{r0})$ ... ...Obivously all sets of this sequence are nonempty, bonded, convex and closed. Apart from this we have that $B^{n+1}_{r0} \subset B^n_{r0} \subset B_{r0}\ for\ n = 1,2,3$ ....Thus, keeping in mind that k<1 and taking into account of eq(3.10),we infer that $\lim_{n\to\infty}\mu(B^n_{r0}) = 0$ Hence. In view of the axiom ($A_6$) of definition 2.1,we deduce that the set $y = \cap_{n=1}^{\infty} B^n_{r0}$ is nonempty, bounded, convex and closed. Moreover, in the light of Remark 2.1, we have that $Y\epsilon ker\mu$. let also observe that the operator V maps the set Y into itself.

**Step II :-**Now we proves that V is the continuous on the set Y.

Let fix $\epsilon > 0$ and take the arbitrary function $x, y\epsilon\ Y$ such that $||x-y|| \le \in$Taking into account the fact that $Y\epsilon ker\mu$ and the description of sets from ker $\mu$ we can find T>0 such that for all $x, y\epsilon\ Y$ and $t \ge T$ we have that$|x(t) - y(t)| \le \in$.

Now assume that $t \geq T$. then keeping in mind that $x, y\epsilon\, Y$ and $v: Y \to Y$,we derive easily the following estimate:

$$\left|(v_x)(t) - (v_y)(t)\right| \leq \in$$

Further, take $t \in [0, T]$.Then applying our assumption and evaluating similarly as above, we obtain

$$(Vx)(t) - (Vy)(t)| \leq p|x(\eta(t) - y(\eta(t))| + \frac{kqa(t)|x(\beta(t) - y(\beta(t)}{\Gamma(\alpha)} \int_0^t \frac{b(s)\phi(|x(\Upsilon(s))|)}{(t-s)^{1-\propto}}$$

$$+ \frac{a(t)[q|y(\beta(t))| + |f(t,0)|]}{\Gamma(\alpha)} \int_0^t \frac{b(s)[|u(s, x(y(s)))|]}{(t-s)^{1-\propto}} ds$$

$$\preccurlyeq p \in + \frac{kqa(t)\bar{b}(t)\Phi(r_0)}{\Gamma(\alpha)} \epsilon + \frac{qa(t)\bar{b}(t)r_0}{\Gamma(\alpha)} \omega^T(u, \epsilon) + \frac{\bar{F}a(t)\bar{b}(t)}{\Gamma(\alpha)} \omega^T(u, \epsilon) \qquad (3.12)$$

Where we denote

$\omega^T(u, \epsilon) = \sup\{|u(t, x) - u(t, y)|: t\epsilon[0, T], |x|, |y| \preccurlyeq \in\}$.

Observe that view of uniform continuity of the function u (t, x)

We have that $\omega^T(u, \epsilon) \to 0\ as\ \in \to 0$.

Now linking Eqs. (3.11) and (3.12),we calculated that the operator V is continuous on the set Y.

Finally, taking into account the properties of the set Y and the operator $V: Y \to Y\ established$ above and applying the classical Schauder fixed point theorem we infer that the operator V has at least one fixed point x= x (t) in Y, Obviously, the function x(t) is a solution of eq.(1.1).Moreover, the solutions are uniformly locally attractive on $R_+$.

## 4) Application

These results can be applied to the following functional integral equation of fractional order with deviating arguments:

$$x(t) = g\left(t, x(\eta_1(t)) \dots \dots . . x(\eta_1(t))\right) + \frac{f(t,x(\beta_1 t)), \dots x(\beta_m(t))}{\Gamma(\alpha)} \int_0^t \frac{u(t,s,x(\gamma_n(s)))}{(t-s)^{1-x}} ds \qquad (4.1)$$

Where $t\epsilon R_1$ and $\alpha$ is a fixed number$\alpha\epsilon(0,1)$.

Assume the following conditions

($B_1$) The function $g: R_- * R^1 \to R$ is continuous and there exists constants $p_i \geq 0 (i = 1,2 \dots . l)$ such that

$$|g(t, x_1 \dots \dots x_1) - g(t, y_1, \dots . . y_1)| \leq \sum_{i=1}^m p_i\, |x_i - y_i| \qquad (4.2)$$

For all $t \in R_+$ and for all $(x_1, x_2, \dots \dots, x_l), (y_1, y_2, \dots \dots, y_l) \in R$.

($B_2$)The function $f: R_- * R^m \to R$is continuous and there exists constants $q_i \geq 0 (i = 1,2 \dots . m)$such that

$$|f(t, x_1 \dots \dots x_m) - g(t, y_1, \dots . . y_m)| \leq \sum_{i=1}^m q_i\, |x_i - y_i| \qquad (4.3)$$

($B_3$) The function $\eta_i, \beta_i, \gamma_i: R_+ \to R_+$ are continuous (i=1,2…….,l ; j=1,2,…..m and k=1,2,……n).

($B_4$)The function $h: R_+ * R_+ \to R_+$is continous and there exists functions$a, b: R_+ \to R_+$ being continuous on $R_+$such that

$$|h(t, s)| \leq a(t)b(s)$$

For any $t, s \in R_+$.

($B_5$)The function $u: R_+ * R_+ \to R_+$ is continuous and Moreover, there exists a function $\Phi: R_+ \to R_+$ being continuous and non decreasing on $R_+$ and constants$K_i \geq 0 (i = 1,2, \dots \dots n)$ such that

$$\left||u(t, x_1, \dots x_n)| \leq \sum_{i=1}^n k_i\, \Phi(|x_i|)\right| \qquad (4.4)$$

For any $t \in R_+$. And for all $s \in R_+$.

Now denote by $\bar{G}$ and$\bar{F}$, the following constants $\bar{G} = \sup\{|f(t, 0, \dots .0)|: t \in R_+\}$,$\bar{F} = \sup\{|f(t, 0, \dots .0)|: t \in R_+\}$. Obviously, $\bar{G}\bar{F} < \infty$ in view of assumptions ($B_1$) and ($B_2$). Further, let us denote by $\bar{b}(t)$ the function defined on $R_+$. in the following way:

$$\bar{b}(t) = \int_0^t \frac{b(s)}{(t-s)^{1-\alpha}} ds.$$

It is easily seen that $\bar{b}(t)$is continuous on $R_+$. The function $\bar{n}: R_+ \to R_+$defined by the formula$\bar{n}(t) = a(t)\bar{b}(t)$, Is bounded on $R_+$ and $\lim_{t\to\infty} \bar{n}(t) = 0$.Keeping in mind, the above assumption we define the following constants:$\bar{N} = \sup\{\bar{n}(t): t \in R_+\}$.In order to formulate our last assumption, let us denote $p = \sum_{i=1}^{l} p_i . q$ and $p = \sum_{i=1}^{m} q_i$ and $k = \sum_{i-1}^{n} k_i$, $N = kq\bar{N}$ and $M = \bar{F}\bar{N}$ There exists a positive solution $r_0$of the inequality

$$pr + \bar{G} + \frac{1}{\Gamma(\alpha)}[Nr\Phi(r) + M\Phi(r)] \preccurlyeq r,$$

Such that$\left(p + \frac{N\Phi(r_0)}{\Gamma(\alpha)}\right) < 1.$

Under the assumptions ($B_1$)-($B_5$) ,it is easy to prove that Eq. (4.1) has at least one solution x=x(t)which belongs to the space BC( $R_+$)and moreover, the solutions are uniformly locally attractive on $R_+$.

**5.Example**

Consider the following nonlinear functional integral equation of fraction order with deviating arguments:

$$x(t) = \frac{t^2+varctg(x(t/2)}{4+5t^2} + \frac{cos^2 t+\frac{x(t/3)}{3+t^2}}{\Gamma(1/2)}\int_0^t \frac{se^{-\delta t} ln\left(1+\sqrt{|\frac{x(s/4)s}{4}}\right)|}{(t-s)^{1/2}} ds, \quad (5.2)$$

Where $t\epsilon R_+$and $v$ is positive constant.Moreover,$\delta$ is fixed natural number.

Notice that the above equation is a special case of Eq.(1.1)if we put$\alpha = \frac{1}{2}, \eta(t) = \frac{t}{2}, \beta(t) = \frac{t}{3}, \Upsilon(t) = t/4$and

$$g(t,x) = \frac{t^2 + varctg(x(t/2)}{4 + 5t^2}$$

$$f(t,x) = cos^2 t + \frac{x}{3+t^2},$$

$$h(t,s) = se^{-\delta t}, u(t,x) = ln(1 + \sqrt{|x|})$$

Obliviously the functions $\eta(t), \beta(t) and\ \gamma(t)$satisfy assumption(H3),In fact, we have that the functions g(t, x)and f(t, x) satisfying assumptions(H1)and(H2)with p=v/4 and q=0.3333 and hence $\bar{G} = 0.2\ and\ \bar{F} = 1$.

Further observe that the assumptions (H4) and (H5) are satisfied with$a(t) = e^{-\delta t}, b(s) = s, \Phi(|x|) = \sqrt{|x|}$ and k=1.

Next we check that the assumption (H6) is satisfied, let us notice that the function $\bar{n}(t)$appearing in that assumption takes the form.$\bar{n}(t) = \frac{4}{3} t^{3/2} e^{-\delta t}$.Thus it is easily seen that assumption (h6) is satisfied and hence we get $\bar{N} = \frac{4}{3e}(\frac{1}{\delta})^{3/2}$ and$M = 0.4905(\frac{1}{\delta})^{3/2}$.

Now let us note that the inequality from (H7) has the form

$$\frac{v}{4}r + 0.2 = \frac{1}{\Gamma(1/2)}\left[0.1635\left(\frac{1}{\delta}\right)^{\frac{3}{2}} {}^{r}\sqrt{r} + 0.4905\left(\frac{1}{\delta}\right)^{\frac{3}{2}}\sqrt{r}\right] \leq r. \quad (5.2)$$

It is easily seen that the number $r_0$=1 is a solution of inequality (5.2) if we take v=1 and δ=1Obiviously the second inequality from assumption (H7) is automatically satisfied in our situation.

Thus on the basis of Theorem 3.1 we conclude that Eq. (5.1) has at least one solution in the space BC($R_+$) belonging to the ball $B_1$ provided v-1 and δ=1. Moreover the solutions are uniformly locally attractive on $R_+$.

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